

The Noncommutative Kähler Geometry of the Standard Podleś Sphere

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Abstract

Building on the now established presentation of the standard Podleś sphere as an example of a noncommutative complex structure, we investigate how its classical Kähler geometry behaves under q -deformation. Discussed are noncommutative versions of Hodge decomposition, Lefschetz decomposition, the Kähler identities, and the refinement of de Rham cohomology by Dolbeault cohomology.

1 Introduction

Since its introduction, the Podleś sphere [18] has served as an example of central importance for many areas of noncommutative geometry. We cite the role it has played in the theory of covariant differential calculi [19, 20, 8], quantum principal bundles [2], quantum frame bundles [13, 12], cyclic cohomology [14, 5], and spectral triples [3, 21].

In recent years, the Podleś sphere has assumed a similarly important role in the newly emerging field of noncommutative complex geometry. The existence for the Podleś sphere of a q -deformed Dolbeault double complex was discovered independently by Majid, and by Heckenberger and Kolb. In [13] it was arrived at using a frame bundle approach, while in [6] it emerged from a classification of the covariant first order differential calculi of the irreducible quantum flag manifolds. A definition of noncommutative complex structure would later be introduced in [9] in order to formalize the properties of this q -Dolbeault complex. A subsequent more comprehensive version of this definition would appear in [1], following which a third version, tailored for quantum homogeneous spaces, was introduced by the author in [16]. Aspects of the noncommutative Hermitian geometry of the Podleś sphere have also been investigated. In [11] and [13] there appeared quantum versions of the 2-sphere's covariant Hermitian metric, along with an associated Hodge operator.

Classically, S_q^2 is not just a Hermitian manifold, but a Kähler manifold. The goal of this paper is to build upon the noncommutative complex and Hermitian constructions

outlined above, and to propose the Podleś sphere as a prototypical example of a non-commutative Kähler structure. To justify this proposal we establish noncommutative versions of Hodge decomposition, Lefschetz decomposition, the Kähler identities, and the refinement of de Rham cohomology by Dolbeault cohomology. An abstract definition for noncommutative Kähler structure will appear in [17], along with general proofs of the results established in this paper, and applications thereof to the quantum projective spaces.

The paper is organised as follows: In section 2 we will recall the definition of the Podleś sphere as a quantum homogeneous space of $C_q[SU_2]$; its quantum Dolbeault double complex; as well as its Hermitian metric and associated Hodge operator.

In section 3, we will show that the Hodge decompositions of $\Omega^1(S^2)$ with respect to d , ∂ , and $\bar{\partial}$ carry over directly to the quantum setting. This allows us to show that the dimensions of the cohomology groups of $\Omega_q^\bullet(S^2)$ have classical lower bounds.

In section 4, we will introduce natural quantum analogues of the Lefschetz and dual Lefschetz operators. Moreover, we will show that these induce a direct generalisation of the Lefschetz decomposition.

In Section 5, we will show that the classical Kähler identities for the two sphere carry over to the quantum setting undeformed. Using this result we can then easily conclude that Dolbeault cohomology is a refinement of de Rham cohomology, generalising another important result of classical Kähler theory.

2 Preliminaries

In this section we fix notation and recall the definitions, constructions, and results from the basic theory of the Podleś sphere. References are provided where proofs or basic details are omitted.

2.1 The Podleś Sphere

In this subsection we will first recall some basic facts about general theory of faithfully flat quantum homogeneous spaces. We will then consider the standard Podleś sphere as an example.

2.1.1 Faithfully Flat Quantum Homogeneous Spaces

Let H, G be two Hopf $*$ -algebras, and let us denote the coproducts, counits, antipodes, and $*$ -maps of both by Δ , ε , S , and $*$ respectively. Moreover, for $\pi : G \rightarrow H$ a Hopf $*$ -algebra map, let us write $\Delta_\pi := (\text{id} \otimes \pi) \circ \Delta$. We denote the coinvariant $*$ -subalgebra of Δ_π by G^H , that is

$$G^H := \{g \in G \mid \Delta_\pi(g) = g \otimes 1\}.$$

We call an algebra of the form G^H a *quantum homogeneous space*. An important fact is that every quantum homogeneous space has a canonical left G -coaction $\Delta_L : M \rightarrow G \otimes M$, induced in the obvious way by the coproduct of G .

We say that G is a *faithfully flat* module over M if the *tensor product functor* $G \otimes_M - : {}_M\mathcal{M} \rightarrow {}_{\mathbf{C}}\mathcal{M}$, from the category of left M -modules to the category of complex vector spaces, preserves and reflects exact sequences.

Let us now explain why faithful flatness is important to us: For a quantum homogeneous $M = G^H$, let \mathcal{E} be an M -bimodule endowed with a left G -coaction Δ_L , satisfying the compatibility condition

$$\Delta_L(mem') = m_{(1)}e_{(-1)}m'_{(1)} \otimes m_{(2)}e_{(0)}m'_{(2)}, \quad (\text{for all } m, m' \in M, e \in \mathcal{E}).$$

Moreover, let us denote by $\Phi(\mathcal{E})$, the right H -comodule $\mathcal{E}/(M^+\mathcal{E})$ with coaction $\Delta_R(\bar{e}) = \overline{e_{(1)}} \otimes \pi(S(e_{(2)}))$. Now if G is a faithfully flat module over M , It follows from a result of Takeuchi [22], that we have an isomorphism

$$\text{frame}_M : \mathcal{E} \rightarrow (G \otimes \Phi(\mathcal{E}))^H, \quad e \mapsto e_{(-1)} \otimes \overline{e_{(0)}}$$

For a more in depth description of this important result, see [16] and references therein.

2.1.2 The Standard Podleś Sphere

In this paper, the quantum homogeneous space we will be working with is the Podleś sphere. Let us begin recalling its definition by recalling the well-known choice of G in this case. For $q \in \mathbf{C}^\times$, the \mathbf{C} -algebra $\mathbf{C}_q[SL_2]$ is generated by the four elements a, b, c, d , subject to the relations

$$\begin{aligned} ab &= qba, & ac &= qca, & bc &= cb, \\ ad - da &= qbc, & ad - qbc &= 1. \end{aligned}$$

It can be given a coalgebra structure with a coproduct uniquely determined by

$$\begin{aligned} \Delta(a) &= a \otimes a + b \otimes c, & \Delta(b) &= a \otimes b + b \otimes d, \\ \Delta(c) &= c \otimes a + d \otimes c, & \Delta(d) &= c \otimes b + d \otimes d, \end{aligned}$$

and a counit uniquely determined by $\varepsilon(a) = \varepsilon(d) = 1$, and $\varepsilon(b) = \varepsilon(c) = 0$. This coalgebra structure is easily seen to be extendable to a Hopf algebra structure. The corresponding antipode S satisfies the relations $S(a) = d$, $S(b) = -q^{-1}c$, $S(c) = -qb$, and $S(d) = a$. Finally, $\mathbf{C}_q[SL_2]$ can be given a Hopf $*$ -algebra structure uniquely determined by $*(a) = d$, $*(b) = -qb$, $*(c) = -q^{-1}c$, and $*(d) = a$. When $\mathbf{C}_q[SL_2]$ is endowed with this $*$ -structure, we denote it by $\mathbf{C}_q[SU_2]$. Note that, for any $f \in \mathbf{C}_q[SU(2)]$, we will use $*(f)$ and f^* interchangeably.

Turning now to our choice for H , we recall that $\mathbf{C}[U_1]$ is the commutative algebra over \mathbf{C} generated by t and t^{-1} , subject to the obvious relation $tt^{-1} = t^{-1}t = 1$. It has a

Hopf $*$ -algebra structure uniquely determined by $\Delta(t) = t \otimes t$; $\varepsilon(t) = 1$; $S(t) = t^{-1}$; and $*(t) = t^{-1}$.

Finally, we come to the question of a map from G to H , and choose $\pi : \mathbf{C}_q[SU(2)] \rightarrow \mathbf{C}[U_1]$ to be the unique Hopf algebra map determined by

$$\pi(a) = t^{-1}, \quad \pi(d) = t, \quad \pi(b) = \pi(c) = 0.$$

We call the corresponding coinvariant subalgebra $\mathbf{C}_q[SU_2]^{\mathbf{C}[U_1]}$ the *Podleś sphere*, and denote it by $\mathbf{C}_q[S^2]$. As a unital algebra it is generated by the elements $b_- := ab, b_0 := bc, b_+ := cd$. Moreover, $\mathbf{C}_q[SU_2]$ is a faithfully flat module over $\mathbf{C}_q[S^2]$ [15].

We should note that the coaction Δ_π induces a \mathbf{Z} -grading on $\mathbf{C}_q[SU_2]$, where, for example, we have

$$\deg(a) = \deg(c) = -1, \quad \deg(b) = \deg(d) = 1.$$

Clearly, $\mathbf{C}_q[S^2]$ is the degree-0 part of $\mathbf{C}_q[SU_2]$. More generally, we denote the degree- k part by \mathcal{E}_k .

2.2 The Noncommutative Complex Geometry of the Podleś Sphere

We will now recall the noncommutative complex geometry of the Podleś sphere referred to in the introduction. We begin by recalling the basic definitions of noncommutative complex geometry in general, and then move on to the specific case of the Podleś sphere.

2.2.1 Noncommutative Complex Structures

A pair (Ω^\bullet, d) is called a *differential algebra* if $\Omega^\bullet = \bigoplus_{k \in \mathbf{N}_0} \Omega^k$ is an \mathbf{N}_0 -graded algebra, and d is a degree 1 map such that $d^2 = 0$, and for which the *graded Leibniz rule* is satisfied

$$d(\omega \wedge \nu) = d\omega \wedge \nu + (-1)^k \omega \wedge d\nu, \quad (\omega \in \Omega^k, \nu \in \Omega^\bullet).$$

A *total differential calculus* over an algebra A is a differential algebra $(\Omega(A), d)$, such that $\Omega^0 = A$, and $\Omega^k = \text{span}_{\mathbf{C}}\{a_0 da_1 \wedge \cdots \wedge da_k \mid a_0, \dots, a_k \in A\}$. We call a differential calculus (Ω^\bullet, d) over a $*$ -algebra A a *total $*$ -differential calculus*, if the involution of A extends to an involutive conjugate-linear map $*$ on Ω^\bullet , for which $(d\omega)^* = d\omega^*$, for all $\omega \in \Omega$, and

$$(\omega_p \omega_q)^* = (-1)^{pq} \omega_q^* \omega_p^*, \quad (\text{for all } \omega_p \in \Omega^p, \omega_q \in \Omega^q).$$

Definition 2.1. An *almost complex structure* for a total $*$ -differential calculus $\Omega^\bullet(A)$ over a $*$ -algebra A , is an \mathbf{N}_0^2 -algebra grading $\bigoplus_{(p,q) \in \mathbf{N}_0^2} \Omega^{(p,q)}$ for $\Omega^\bullet(A)$ such that, for all $(p, q) \in \mathbf{N}_0^2$:

1. $\Omega^k(A) = \bigoplus_{p+q=k} \Omega^{(p,q)}$;

2. the wedge map restricts to isomorphisms

$$\wedge : \Omega^{(p,0)} \otimes_A \Omega^{(0,q)} \rightarrow \Omega^{(p,q)}, \quad \wedge : \Omega^{(0,q)} \otimes_A \Omega^{(p,0)} \rightarrow \Omega^{(p,q)}; \quad (1)$$

3. $*(\Omega^{(p,q)}) = \Omega^{(q,p)}$.

We call an element of $\Omega^{(p,q)}$ a (p, q) -form.

We say that a total differential calculus $\Omega^\bullet(M)$ over a quantum homogeneous space $M = G^H$ is *covariant* if the coaction Δ_L extends to a left coaction on $\Omega^\bullet(M)$ such that

$$\Delta_L \circ d = (\text{id} \otimes d) \circ \Delta_L.$$

Moreover, we say that an almost complex structure $\Omega^\bullet(M)$ is *left-covariant* if we have

$$\Delta_L(\Omega^{(p,q)}) \subseteq G \otimes \Omega^{(p,q)}, \quad (\text{for all } (p, q) \in \mathbf{N}^2).$$

Directly generalising the classical definition, we say that an almost-complex structure $\Omega^{(\bullet, \bullet)}$ is *integrable* if $d(\Omega^{(1,0)}) \subseteq \Omega^{(2,0)} \oplus \Omega^{(1,1)}$, or equivalently, if $d(\Omega^{(0,1)}) \subseteq \Omega^{(1,1)} \oplus \Omega^{(0,2)}$. The assumption of integrability has some very useful consequences.

Lemma 2.2 *If an almost complex structure $\bigoplus_{(p,q) \in \mathbf{N}_0^2} \Omega^{(p,q)}$ is integrable, then*

1. $d = \partial + \bar{\partial}$;
2. $(\bigoplus_{(p,q) \in \mathbf{N}^2} \Omega^{(p,q)}, \partial, \bar{\partial})$ is a double complex;
3. $\partial(a^*) = (\bar{\partial}a)^*$, and $\bar{\partial}(a^*) = (\partial a)^*$, for all $a \in A$;
4. both ∂ and $\bar{\partial}$ satisfy the graded Leibniz rule.

Finally, we come to the definition of cohomology groups: Just as in the classical case, the *de Rham cohomology groups* of $\Omega_q^\bullet(S^2)$ are $H^k(S^2) = \ker(d|_{\Omega^k}) / (\text{im } d|_{\Omega^{k-1}})$. Moreover, the *holomorphic* and *anti-holomorphic Dolbeault cohomology groups* are respectively defined by

$$H_\partial^{(a,b)}(S^2) = \ker(\partial|_{\Omega^{a,b}}) / \text{im}(\partial|_{\Omega^{a-1,b}}), \quad H_{\bar{\partial}}^{(a,b)}(S^2) = \ker(\bar{\partial}|_{\Omega^{a,b}}) / \text{im}(\bar{\partial}|_{\Omega^{a,b-1}}).$$

2.2.2 A Noncommutative Complex Structure for the Podleś Sphere

In the case of the Podleś sphere, there exists only one covariant total differential calculus $\Omega_q^\bullet(S^2)$ whose 1-forms have classical dimension and which admits a covariant almost complex structure. Moreover, $\Omega_q^\bullet(S^2)$ admits only one such almost complex structure, and it is integrable.

While we will not give a complete description of the calculus here, we will need to recall some important facts. Firstly, let us denote

$$V^{(1,0)} := \Phi(\Omega^{(1,0)}), \quad V^{(0,1)} := \Phi(\Omega^{(0,1)}).$$

Both $V^{(1,0)}$ and $V^{(0,1)}$ are 1-dimensional. We choose a basis for both according to

$$e^+ := \overline{db_+}, \quad e^- := \overline{db_-}.$$

The right $\mathbf{C}[U_1]$ -comodule structures of each are determined by $\Delta_R(e^\pm) = e^\pm \otimes t^{\pm 2}$. This immediately implies that

$$\Omega^{(1,0)} = \mathcal{E}_2 \otimes e^+, \quad \Omega^{(0,1)} = \mathcal{E}_{-2} \otimes e^-.$$

Now $\Phi(\Omega_q^2(S^2))$ is a 1-dimensional vector space, for which we choose $\tau := \overline{db_+ \wedge db_-}$ as a basis. It is easily shown that $\Delta_R(\tau) = \tau \otimes 1$, and so we have that

$$\Omega_q^2(S^2) \simeq \mathbf{C}_q[S^2] \otimes \tau.$$

Finally, for all $k > 2$, we have that $\Omega_q^k(S^2) = 0$. For $f \in \mathcal{E}$, $g \in \mathcal{E}$, the multiplication \wedge in $\Omega_q^\bullet(S^2)$ is uniquely determined by $(f \otimes e^\pm) \wedge (g \otimes e^\pm) = 0$, and

$$(f \otimes e^+) \wedge (f \otimes e^-) = fg\tau, \quad (g \otimes e^-) \wedge (g \otimes e^+) = -q^2 g f \tau.$$

2.3 The Noncommutative Hermitian Geometry of the Podleś Sphere

Let us start this subsection by constructing a sesqui-linear map

$$g : \Omega_q^1(S^2) \otimes_{\mathbf{C}_q[S^2]} \Omega_q^1(S^2) \rightarrow \mathbf{C}_q[S^2],$$

which we call the *metric* of $\Omega_q^1(S^2)$: For $f \in \mathcal{E}_2$, and $h \in \mathcal{E}_{-2}$, we define g to be the unique sesqui-linear mapping for which

$$g(fe^+ \otimes he^-) = fh^*, \quad g(he^- \otimes fe^+) = q^{\pm 2} hf^*, \quad g(fe^+ \otimes fe^+) = g(he^- \otimes he^-) = 0.$$

The fact that there are no zero divisors in $\mathbf{C}_q[SU(2)]$ clearly implies that the map is non-degenerate. Note that since fh^* and h^*f are both elements of degree 0 with respect to the \mathbf{Z} -grading on $\mathbf{C}_q[SU_2]$, the image of g does indeed lie in $\mathbf{C}_q[S^2]$.

Let us now consider the element

$$\mathbf{g} := e^+ \otimes e^- + q^2 e^- \otimes e^+ \in \Omega_q^1(S^2) \otimes \Omega_q^1(S^2).$$

It was first considered by Majid in [13], as a q -deformation of the standard metric on the two sphere. It has the important property that $\wedge(\mathbf{g}) = 0$, which can be considered as a q -deformation of the symmetry of the metric. As is easy to see, \mathbf{g} is the unique element of $\Omega_q^1(S^2) \otimes \Omega_q^1(S^2)$ that satisfies the identity

$$(\text{id}, g(\cdot, \omega))\mathbf{g} = \omega, \quad (g(\cdot, \omega), \text{id})\mathbf{g} = \omega, \quad (\text{for all } \omega \in \Omega_q^1(S^2)).$$

Motivated by the terminology of [13], we call \mathfrak{g} the *inverse* of \mathfrak{g} . It will prove important for the definition of the fundamental form of \mathfrak{g} .

Now we can easily extend g to a map from $\Omega_q^\bullet(S^2) \otimes_{\mathbf{C}_q[S^2]} \Omega_q^\bullet(S^2)$ to $\mathbf{C}_q[S^2]$: For any $f, g \in \mathbf{C}_q[S^2]$, we set

$$g(f, g) = fg^*, \quad g(fe^+ \wedge e^-, ge^+ \wedge e^-) = fg^*,$$

and moreover require $\mathbf{C}_q[S^2]$, $\Omega_q^1(S^2)$, and $\Omega_q^2(S^2)$ to be orthogonal with respect to g .

A *Hodge map* $*_H$ associated to the metric $(\langle \cdot, \cdot \rangle)$, was introduced in [10]. It is defined to be the unique map $*_H : \Omega^\bullet[\mathbf{C}P^1] \rightarrow \Omega^\bullet[\mathbf{C}P^1]$, for which

$$\langle \omega, \nu \rangle \tau = \omega \wedge (*_H(\nu)).$$

As is easily seen, an explicit description of $*$ is given by

$$*_H(1) = \tau, \quad *_H(\tau) = 1, \quad *_H(e^+) = ie^+, \quad *_H(e^-) = -ie^-.$$

As an elementary calculation will verify, $*_H$ commutes with $*$.

3 The Hodge Decompositions

We will now turn to the question of how to calculate the various cohomology groups of $\Omega_q^\bullet(S^2)$. Classically, this is most easily done using Hodge theory, and we shall follow a similar path here. We begin by considering the question of adjoint operators for d , ∂ , and $\bar{\partial}$:

Lemma 3.1 *The functional \int is closed.*

Proof. Any $\omega \in \Omega^{(1,0)}$ can be rewritten as $f \otimes e^+$, for some $f \in \mathcal{E}_{-2}$. We then have that $d\omega = d(f \otimes e^+) = f_{(1)} \otimes \overline{(f_{(2)})^+} \wedge e^+$. Thus, we see that

$$\int d(f \otimes e^+) = \hat{\tau}(h(f_{(1)}), \overline{f_{(2)}^+} \wedge e^+) = \overline{h(f)^+} \wedge e^+ = 0$$

The proof for $\omega' \in \Omega^{(0,1)}$ is exactly analogous. \square

Lemma 3.2 *With respect to the inner product $\langle \cdot, \cdot \rangle$, the operators d , ∂ , and $\bar{\partial}$ are adjointable, with explicit formulae being given by*

$$d^* = - * d *, \quad \partial^* = - * \bar{\partial} *, \quad \bar{\partial}^* = - * \partial *.$$

Proof. The proofs for the three operators are a direct generalisation of the classical proof. For completeness, we present the case for ∂ : For $\omega \in \Omega_q^k(S^2)$, $\nu \in \Omega_q^{k+1}(S^2)$, with $k = 0, 1$, we have that

$$\langle \omega, \partial^* \nu \rangle = - \int g(\omega, *_H \bar{\partial} *_H \nu) \tau = - \int \omega \wedge (*_H^2(\bar{\partial} *_H \nu)^*) = (-1)^{k+1} \int \omega \wedge (\partial *_H \nu^*).$$

Now the fact that $\int d = 0$, easily implies that $\int \partial = 0$, and so,

$$\int \partial \omega \wedge *_H \nu^* = (-1)^{k+1} \int \omega \wedge \partial *_H \nu^*.$$

This tells us that

$$\langle \omega, \partial^* \nu \rangle = \int \partial \omega \wedge *_H \nu^* = \int g(\partial \omega, \nu) \tau = \langle \partial \omega, \nu \rangle,$$

as required. \square

We call the operator adjoints of d , ∂ , and $\bar{\partial}$, the *codifferential*, *holomorphic codifferential*, and *anti-holomorphic codifferential*, respectively. Using these operators we can introduce q -versions of the classical Dirac and Laplace operators according to

$$D_d = d + d^*, \quad D_\partial = \partial + \partial^*, \quad D_{\bar{\partial}} = \bar{\partial} + \bar{\partial}^*,$$

and $\Delta_d = D_d^2$, $\Delta_\partial = D_\partial^2$, and $\Delta_{\bar{\partial}} = D_{\bar{\partial}}^2$.

Lemma 3.3 *The Hodge map $*_H$ commutes with the Laplacians Δ_d , Δ_∂ , and $\Delta_{\bar{\partial}}$.*

Proof. The proofs are again direct generalisations of the classical versions. For sake of completeness, we present the case for ∂ :

$$\begin{aligned} *_H \circ \Delta_\partial &= - *_H \partial *_H \bar{\partial} *_H - *_H^2 \bar{\partial} *_H \partial = - *_H \partial *_H \bar{\partial} *_H - \bar{\partial} *_H \partial *_H^2 \\ &= (- *_H \partial *_H \bar{\partial} *_H - \bar{\partial} *_H \partial *_H) *_H = \Delta_\partial *_H \end{aligned}$$

\square

Let us recall the standard basis for $\mathbf{C}_q[SU(2)]$ $\{a^i b^j c^k \mid i, j, k \in \mathbf{N}_0\}$. Using it we can define $\mathbf{C}_q[SU(2)]_k$, for $k \in \mathbf{N}_0$, as

$$\mathbf{C}_q[SU(2)]_k := \text{span}_{\mathbf{C}} \{a^i b^j c^k \mid i + j + k = n\}$$

and note that each $\mathbf{C}_q[SU(2)]_k$ is a sub-coalgebra of $\mathbf{C}_q[SU(2)]$. We then denote

$$\Omega_k := (\mathbf{C}_q[SU(2)]_k \otimes V^\bullet)^{U_1}$$

Lemma 3.4 *Each $\Omega^1(S^2)_k$ is a left $\mathbf{C}_q[SU_2]$ -comodule, and we have the decomposition*

$$\Omega_q^\bullet(S^2) = \bigoplus_{k \in \mathbf{N}_0} \Omega_k. \quad (2)$$

Moreover, both d and d^* restrict to linear endomorphisms of Ω_k , for each $k \in \mathbf{N}_0$.

Proof. The fact that each $\mathbf{C}_q[SU(2)]_k$ is a coalgebra directly implies that $\Omega^1(S^2)_k$ is a left $\mathbf{C}_q[SU_2]$ -comodule, as a moments thought will confirm. Now let $\sum_i f_i \otimes v_i$ be an element of $\Omega_q^\bullet(S^2)$, such that $f_i \in \Omega_i$, for all i , and

$$\Delta_R(\sum_i f_i \otimes v_i) = \sum_i f_i \otimes v_i \otimes 1.$$

Then the fact that $\Omega^1(S^2)_k$ is a left $\mathbf{C}_q[SU_2]$ -comodule implies that

$$\Delta_R(f_i \otimes v_i) = f_i \otimes v_i \otimes 1, \quad \text{for all } i.$$

The decomposition in (3.4) now follows directly. \square

With this lemma in hand, we can now move onto the main result of this section.

Theorem 3.5 *For $\Omega_q^1(\Omega^\bullet(S^2))$, it holds that:*

1. *The Dirac operators D_d, D_∂ , and $D_{\bar{\partial}}$, as well as the Laplacians $\Delta_d, \Delta_\partial$, and $\Delta_{\bar{\partial}}$, are diagonalisable;*
2. *It holds that*
 - (a) $\ker(\Delta_d) = \ker(D_d) = \ker(d) \cap \ker(d^*)$,
 - (b) $\ker(\Delta_\partial) = \ker(D_\partial) = \ker(\partial) \cap \ker(\partial^*)$,
 - (c) $\ker(\Delta_{\bar{\partial}}) = \ker(D_{\bar{\partial}}) = \ker(\bar{\partial}) \cap \ker(\bar{\partial}^*)$;
3. *We have the three decompositions*
 - (a) $\Omega_q^\bullet(S^2) = \mathcal{H}_d \oplus d(\Omega_q^\bullet(S^2)) \oplus d^*(\Omega_q^\bullet(S^2))$,
 - (b) $\Omega_q^\bullet(S^2) = \mathcal{H}_\partial \oplus \partial(\Omega_q^\bullet(S^2)) \oplus \partial^*(\Omega_q^\bullet(S^2))$,
 - (c) $\Omega_q^\bullet(S^2) = \mathcal{H}_{\bar{\partial}} \oplus \bar{\partial}(\Omega_q^\bullet(S^2)) \oplus \bar{\partial}^*(\Omega_q^\bullet(S^2))$.

Proof. Let us denote by $d_k, \partial_k, \bar{\partial}, d_k^*, \partial_k^*$, and $\bar{\partial}_k^*$ the respective restrictions to Ω_k of $d, \partial, \bar{\partial}, d^*, \partial^*$, and $\bar{\partial}^*$. It is clear that d_k^*, ∂_k^* , and $\bar{\partial}_k^*$ are the adjoints of d_k, ∂_k , and $\bar{\partial}_k$ respectively. Since

$$D_{d,k} = d_k + d_k^*, \quad D_{\partial,k} = \partial_k + \partial_k^*, \quad D_{\bar{\partial},k} = \bar{\partial}_k + \bar{\partial}_k^*,$$

are all self adjoint operators on a finite dimensional vector space, they are diagonalisable. This immediately implies that that

$$\ker(\Delta_{d,k}) = \ker(D_{d,k}), \quad \ker(\Delta_{\partial,k}) = \ker(D_{\partial,k}), \quad \ker(\Delta_{\bar{\partial},k}) = \ker(D_{\bar{\partial},k}).$$

Now since $\langle d\omega, d^*\omega \rangle = \langle d^2\omega, \nu \rangle = 0$, the spaces $d(\Omega)$ and $d^*(\Omega)$ are orthogonal. Hence, if $D_d(\omega) = 0$, then $d\omega = 0 = d^*\omega$. It follows that $\ker(D) = \ker(d) \cap \ker(d^*)$. That the other versions of the proposition follow is established similarly, and so, we are finished with parts 1 and 2.

Since $\Omega_q(S^2)_k$ is a finite dimensional space, we can choose a subspace $X_k \subseteq \Omega_q(S^2)_k$ such that

$$\Omega_q^\bullet(S^2) = X_k \oplus d(\Omega_q^\bullet(S^2)) \oplus d^*(\Omega_q^\bullet(S^2))$$

is an orthogonal decomposition. Now for any $x \in X_k, \omega \in \Omega_q^\bullet(S^2)$, it must hold that

$$0 = \langle x, d\omega \rangle = \langle d^*x, \omega \rangle, \quad \text{and} \quad 0 = \langle x, d^*\omega \rangle = \langle dx, \omega \rangle.$$

Hence, we must have that $x \in \ker(d) \cap \ker(d^*) = \mathcal{H}_{d,k}$, which tells us that $X_k \subseteq \mathcal{H}_{d,k}$. The opposite inclusion $\mathcal{H}_d \subseteq X_k$ would follow from the orthogonality of \mathcal{H}_d and $d(\Omega_q^\bullet(S^2)) \oplus d^*(\Omega_q^\bullet(S^2))$. But this is directly implied by

$$\langle d\omega + d^*\nu, \mu \rangle = \langle d\omega, \mu \rangle + \langle d^*\nu, \mu \rangle = 0, \quad (\omega, \nu \in \Omega_q^\bullet(S^2), \mu \in \mathcal{H}_d).$$

□

We call the three decompositions given above *Hodge decomposition*, the *holomorphic Hodge decomposition*, and the *anti-holomorphic Hodge decomposition* respectively. As an easy consequence of the theorem we have following important result.

Corollary 3.6 *It holds that*

$$\begin{aligned} \ker(d|_{\Omega^k}) &= \mathcal{H}_d^k \oplus d(\Omega_q^{k-1}(S^2)) \oplus d^*(\Omega_q^{k+1}(S^2)), \\ \ker(\partial|_{\Omega^k}) &= \mathcal{H}_\partial^k \oplus d(\Omega_q^{k-1}(S^2)) \oplus d^*(\Omega_q^{k+1}(S^2)), \\ \ker(\bar{\partial}|_{\Omega^k}) &= \mathcal{H}_{\bar{\partial}}^k \oplus d(\Omega_q^{k-1}(S^2)) \oplus d^*(\Omega_q^{k+1}(S^2)). \end{aligned}$$

and so, we have the following isomorphisms

$$H_d^k \simeq \mathcal{H}_d^k, \quad H_\partial^{(a,b)} \simeq \mathcal{H}_d^{(a,b)}, \quad H_{\bar{\partial}}^{(a,b)} \simeq \mathcal{H}_d^{(a,b)}.$$

Proof. First note that, since $\mathcal{H}_d = \ker(d) \cap \ker(d^*)$, we cannot have $d\omega = 0$, for any $\omega \in d^*(\Omega_q^\bullet(S^2))$. This means that

$$\ker(d|_{\Omega^k}) = \mathcal{H}_d^k \oplus d(\Omega_q^{k-1}(S^2)),$$

which immediately implies that $H_d^k \simeq \mathcal{H}_d^k$. The corresponding isomorphisms for ∂ and $\bar{\partial}$ are established analogously.

□

With this result in hand, we can do some cohomological calculations:

Corollary 3.7 *It holds that*

1. $H^0 \simeq H^2$, with each having dimension greater than or equal to 1;
2. $H_\partial^{(1,0)} \simeq H_\partial^{(0,1)}$, and $H_{\bar{\partial}}^{(1,0)} \simeq H_{\bar{\partial}}^{(0,1)}$

Proof. The fact that $H^0 \simeq H^2$ follows directly from the theorem and the fact that the Laplacians commute with Hodge operator. That H^0 has dimension greater than or equal to 1 follows from the basic identity $\Delta_d(1) = 0$.

The two isomorphisms in the second part of the theorem are a consequence of the fact that the Laplacians commute with the $*$ -operator. \square

4 The Lefschetz Decomposition of $\mathbf{C}_q[\mathbf{CP}^1]$

A major result in the theory of classical Hermitian manifolds is the Lefschetz decomposition. In this section we will formulate a q -deformation of the Lefschetz decomposition of the two-sphere.

Define the *fundamental form* of the metric g , using its inverse \mathbf{g} , to be the 2-form

$$\kappa := (\text{id} \otimes J) \circ \wedge(\mathbf{g}) = -e^+ \wedge e^-,$$

where J is the almost-complex structure map introduced in the previous section. It is important to note that κ is a covariant element of $\Omega_q^1(S^2)$. We then define the Lefschetz operator

$$L : \Omega^\bullet(\mathbf{CP}^1) \rightarrow \Omega^\bullet(\mathbf{CP}^1), \quad \omega \mapsto \omega \wedge \kappa.$$

This operator is covariant due to the fact that κ is coinvariant.

Classically, the covariance of the Lefschetz operator immediately implies that it is adjointable with respect to the metric $\langle \cdot, \cdot \rangle$. The following lemma shows that this carries over to the quantum setting:

Lemma 4.1 *The operator L has an adjoint Λ , which we call the dual Lefschetz operator. Moreover, in direct generalisation of the well-known classical result, an explicit description of Λ is given by $\Lambda = *^{-1} \circ L \circ *$.*

Proof. As is easy to see, the only identities which do not hold trivially are

$$\langle Lf, ke^+ \wedge e^- \rangle = \langle (f, L(ke^+ \wedge e^-)) \rangle, \quad \text{and} \quad \langle L(fe^+ \wedge e^-), k \rangle = \langle fe^+ \wedge e^-, Lk \rangle,$$

where h is of course another element of $\mathbf{C}_q[S^2]$. The first identity follows from

$$\langle Lf, he^+ \wedge e^- \rangle = \int fh^* = \int g(f, \Lambda(he^+ \wedge e^-)),$$

while the second is established similarly. \square

Finally, we introduce the *counting operator* on $\Omega_q^\bullet(S^2)$, defined by setting

$$H : \Omega_q^\bullet(S^2) \rightarrow \Omega_q^\bullet(S^2), \quad v \mapsto \sum_{k=0}^2 (k-1) \Pi^k,$$

where Π^k is the projection onto $\Omega_q^k(S^2)$. The following lemma relates the counting operator with L and Λ in a direct generalisation of a well known classical result:

Lemma 4.2 *It holds that*

$$[H, L] = 2L, \quad [H, \Lambda] = -2\Lambda, \quad [L, \Lambda] = H,$$

and hence, that the operators L, Λ , and H define a representation of \mathfrak{sl}_2 .

Proof. We begin by noting that, for any $\omega \in \Omega_q^k(S^2)$, we have

$$[H, L](\omega) = (k+1)(\kappa \wedge \omega) - \kappa \wedge ((k-1)\omega) = 2\kappa \wedge \omega,$$

which clearly implies the first identity. The second identity follows analogously from

$$[H, \Lambda](\omega) = (k-3)\Lambda(\omega) - (k-1)\Lambda(\omega) = -2\Lambda(\omega).$$

The third identity is most easily established in a case by case manner: For $f \in \mathbf{C}_q[S^2]$, we have

$$[L, \Lambda]f = L \circ \Lambda(f) - \Lambda \circ L(f) = -f = H(f).$$

For $\nu \in \Omega_q^1(S^2)$, we have

$$[L, \Lambda]\nu = L \circ \Lambda(\nu) - \Lambda \circ L(\nu) = \nu = H(\nu).$$

While for $\nu' \in \Omega^2(S^2)$, we have

$$[L, H]\nu' = L \circ \Lambda(\nu') - \Lambda \circ L(\nu') = 0 = H(\nu').$$

□

It is interesting to note that this representation of \mathfrak{sl}_2 splits into a direct sum of subrepresentations according to

$$\Omega_q^\bullet(S^2) = (\mathbf{C}_q[S^2] \oplus \Omega_q^2(S^2)) \bigoplus \Omega_q^1(S^2).$$

This is a q -deformation of the classical Lefschetz decomposition of the de Rham complex of the two-sphere.

5 The Kähler Identities

A standard result of Kähler geometry is that, up to scalar multiple, these three Laplacians coincide classically. We will now show that this fact carries over to the noncommutative setting by following the standard classical proof based on the Kähler identities.

Proposition 5.1 *We have the following relations:*

$$\begin{aligned} [L, \partial^*] &= i\bar{\partial}, & [L, \bar{\partial}^*] &= -i\partial, & [L, \partial] &= 0, & [L, \bar{\partial}] &= 0, \\ [\Lambda, \partial] &= i\bar{\partial}^*, & [\Lambda, \bar{\partial}] &= -i\partial^*, & [\Lambda, \partial^*] &= 0, & [\Lambda, \bar{\partial}^*] &= 0, \end{aligned}$$

Proof. The relations

$$[L, \partial] = [L, \bar{\partial}] = [\Lambda, \partial^*] = [\Lambda, \bar{\partial}^*] = 0$$

are direct consequences of the definition of L and Λ . The remaining relations are easily verified by direct calculation. We show this for the relation $[L, \partial^*] = i\bar{\partial}$: First we note that $[L, \partial^*]$ has a non-zero action only on $\mathbf{C}_q[\mathbf{CP}^1]$ and $\Omega^{(1,0)}(\mathbf{CP}^1)$. For $f \in \mathbf{C}_q[\mathbf{CP}^1]$, we have

$$[L, \partial^*]f = (-L \circ * \bar{\partial} * + * \bar{\partial} * \circ L)f = * \bar{\partial} * \circ Lf = * \bar{\partial} * (f\kappa) = * \bar{\partial} f = i\bar{\partial} f.$$

While for $f\partial h \in \Omega^{(1,0)}(\mathbf{CP}^1)$, we have

$$\begin{aligned} [L, \partial^*](f\partial h) &= (-L \circ * \bar{\partial} * + * \bar{\partial} * \circ L)f\partial h = -L \circ * \bar{\partial} * f\partial h \\ &= -iL \circ * \bar{\partial}(f\partial h). \end{aligned}$$

Now $\bar{\partial}(f\partial h) = ke^+ \wedge e^-$, for some $k \in \mathbf{C}_q[\mathbf{CP}^1]$, and so,

$$[L, \partial^*](f\partial h) = -iL \circ *(ke^+ \wedge e^-) - iL(k) = -ik\kappa = ie^+ \wedge e^-.$$

On the other hand

$$i\bar{\partial}(f\partial h) = ike^+ \wedge e^-,$$

which establishes the relation. □

Corollary 5.2 *Denoting by (\cdot, \cdot) the usual anti-commutator bracket, it holds that*

$$(\partial, \bar{\partial}^*) = (\bar{\partial}, \partial^*) = 0, \quad (\partial, \partial^*) = (\bar{\partial}, \bar{\partial}^*),$$

from which follows the identity $\Delta = 2\Delta_\partial = 2\Delta_{\bar{\partial}}$.

Proof. First note that

$$-i(\bar{\partial}\partial^* + \partial^*\bar{\partial}) = \bar{\partial}[\Lambda, \bar{\partial}] + [\Lambda, \bar{\partial}]\bar{\partial} = \bar{\partial}\Lambda\bar{\partial} - \bar{\partial}\Lambda\bar{\partial} = 0,$$

and similarly $\partial\bar{\partial}^* + \bar{\partial}^*\partial = 0$. This gives that

$$\begin{aligned} \Delta &= (\partial + \bar{\partial})(\partial^* + \bar{\partial}^*) + (\partial^* + \bar{\partial}^*)(\partial + \bar{\partial}) \\ &= (\partial\partial^* + \partial^*\partial) + (\bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}) + (\bar{\partial}\partial^* + \partial^*\bar{\partial}) + (\partial\bar{\partial}^* + \bar{\partial}^*\partial) \\ &= \Delta_\partial + \Delta_{\bar{\partial}}. \end{aligned}$$

It remains to show that $\Delta_\partial = \Delta_{\bar{\partial}}$, which is an easy consequence of the proposition:

$$\begin{aligned} -i\Delta_\partial &= -i(\partial\partial^* + \partial^*\partial) = \partial[\Lambda, \bar{\partial}] + [\Lambda, \bar{\partial}]\partial = \partial\Lambda\bar{\partial} - \partial\bar{\partial}\Lambda + \lambda\bar{\partial}\partial - \bar{\partial}\Lambda\partial \\ &= \partial\Lambda\bar{\partial} + \bar{\partial}\partial\Lambda - \bar{\partial}\Lambda\partial = [\partial, \Lambda]\bar{\partial} + \bar{\partial}[\partial, \Lambda] = -i\bar{\partial}^*\bar{\partial} - i\bar{\partial}\bar{\partial}^* = -i\Delta_{\bar{\partial}}. \end{aligned}$$

□

Taken together with Hodge decomposition, this in turn directly implies the following result. It tells us that, just as in the classical case, the Dolbeault cohomology of $\Omega_q^\bullet(S^2)$ is a refinement of its de Rham cohomology.

Corollary 5.3 *It holds that*

$$H^k = \bigoplus_{a+b=k} H_\partial^{(a,b)} = \bigoplus_{a+b=k} H_{\bar{\partial}}^{(a,b)}.$$

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